A DISCRETE ANALOGUE OF PERIODIC DELTA BOSE GAS AND AFFINE HECKE ALGEBRA

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Abstract. We consider an eigenvalue problem for a discrete analogue of the Hamiltonian of the non-ideal Bose gas with delta-potentials on a circle. It is a two-parameter deformation of the discrete Hamiltonian for joint moments of the partition function of the O'Connell-Yor semi-discrete polymer. We construct the propagation operator by using integral-reflection operators, which give a representation of the affine Hecke algebra. We also construct eigenfunctions by means of the Bethe ansatz method.

1. Introduction

In this paper we study a discrete analogue of the Hamiltonian of the non-ideal Bose gas with delta-potential interactions, which we call the delta Bose gas for short. The eigenvalue problem for the delta Bose gas with periodic boundary condition was solved by Lieb and Liniger by means of the Bethe ansatz method [13]. The Hamiltonian of the system is given by

$$-\Delta+\sum_{\substack{1\le i<j\le k\\ m\in\mathbb{Z}}}\delta(x_i-x_j+m). \tag{1.1}$$

The potential is supported by affine hyperplanes associated with the affine root system of type $A_{k-1}^{(1)}$. Gutkin and Sutherland generalized the periodic delta Bose gas for all any affine root systems [10].

In [11] Heckman and Opdam studied the root system generalization of the delta Bose gas on a line, and revealed a connection with harmonic analysis on homogeneous spaces of semisimple groups. A key observation is that the propagation operators give a representation of the degenerate (or graded) Hecke algebra [7, 14]. For the periodic case, Emsiz, Opdam and Stokman [8] found that the underlying symmetry is governed by Cherednik's degenerate double affine Hecke algebra [3].

In this paper we consider a discrete analogue of the Hamiltonian (1.1). An integrable discretization of the delta Bose-gas has been already proposed and studied by van Diejen [5]. The discrete version which we will consider is different from it and has an origin in the study of integrable stochastic models. The Kardar-Parisi-Zahng (KPZ) equation is a stochastic partial differential equation for height function $\mathcal{H}$ of growing interfaces (see the review [4] for details). The Cole-Hopf solution $\mathcal{Z}:=\exp(-\mathcal{H})$ satisfies the stochastic heat equation. The fact is that the n-th moment of $\mathcal{Z}$ satisfies an evolution equation with the Hamiltonian of the delta

Bose gas on a line with n particles. An integrable discretization of the KPZ equation is the q-deformed totally asymmetric simple exclusion process (q-TASEP) (see [1], Section 3.3.2). In a scaling limit q-TASEP goes to the O'Connell-Yor semi-discrete directed polymer [15]. The joint moment $\tilde{v}(\tau;\overrightarrow{n})\,(\tau\in\mathbb{R}_{>0},\overrightarrow{n}\in(\mathbb{Z}_{\ge 0})^k)$ of its partition function satisfies the following evolution equation [2]:

$$\frac{d}{d\tau}\tilde{v}(\tau;\overrightarrow{n})=\tilde{H}\tilde{v}(\tau;\overrightarrow{n}),\quad \tilde{H}:=\sum_{i=1}^{k}\nabla_i+\sum_{1\le i<j\le k}\delta_{n_i,n_j}, \tag{1.2}$$

where ∇_i is the difference operator

$$(\nabla_i f)(\overrightarrow{n}):=f(n_1,\ldots,n_i-1,\ldots,n_k)-f(n_1,\ldots,n_i,\ldots,n_k).$$

In this paper we consider a two-parameter deformation of the Hamiltonian $\tilde{H}$ with periodic boundary condition. It acts on the space of $\mathbb{C}$-valued functions on the k-dimensional lattice $X=\oplus_{i=1}^{k}\mathbb{Z}v_i$ and is given by

$$H:=\sum_{i=1}^{k}\beta^{d_i^{+}}(t_{v_i}-\alpha d_i^{-}),$$

where $\alpha\in\mathbb{C}$ and $\beta\in\mathbb{C}^{\times}$ are parameters, and t_{v_i} is the shift operator $(t_{v_i}f)(x):=f(x-v_i)$. The functions $d_i^{\pm}$ count positive roots of type $A_{k-1}^{(1)}$ whose values at x are non-positive multiple of the system size (see (3.1) and (3.2) below). Setting $\beta=1$ and $\alpha=-1$ we recover $\tilde{H}$ with periodic boundary condition up to an additive constant.

The main result of this paper is construction of the propagation operator G which sends an eigenfunction of "half Laplacian" $\sum_{i=1}^{k}t_{v_i}$ to that of the Hamiltonian H with the same eigenvalue (see Theorem 5.1 below). To define G we make use of a discrete analogue of the integral-reflection operators due to Yang [16] for the case of type A and Gutkin [9] for the general case. Emsiz and van Diejen [6] constructed the discrete version from a polynomial representation of the affine Hecke algebra. We follow their construction but start from more general divided difference operators satisfying the braid relations, which are classified by Lascoux and Schützenberger [12]. Then our integral-reflection operators also give a representation of the affine Hecke algebra of type GL_k.

The paper is organized as follows. In Section 2 we prepare some notation and lemmas about the affine root system of type $A_{k-1}^{(1)}$. We define the operator H in Section 3 and the associated integral-reflection operators in Section 4. In Section 5 we prove the main theorem. In the last Section 6 we construct Bethe wave functions, which are symmetric and periodic eigenfunctions for the Hamiltonian H parameterized by solutions to a system of algebraic equations.

2. Preliminaries

Throughout this paper we fix two integers $k \geq 2$ and $L \geq 1$. Let V be the k-dimensional Euclidean space with inner product $\langle\cdot,\cdot\rangle$, and V^* the linear dual of V. We also write $\langle\cdot,\cdot\rangle$ for the associated inner product on V^*. Fix an orthogonal basis $\{v_i\}_{i=1}^k$ of V, and set

$$X := \oplus_{i=1}^k \mathbb{Z} v_i.$$

We denote by $\{\epsilon_i\}_{i=1}^k$ the dual basis corresponding to $\{v_i\}_{i=1}^k$.

For $\xi \in V^* \setminus \{0\}$ we define the co-vector $\xi^\vee \in V$ by the property

$$\eta(\xi^\vee) = 2\frac{\langle \eta, \xi\rangle}{\langle \xi, \xi\rangle} \qquad (\forall \eta \in V^*).$$

Let $\mathrm{Aff}(V) := V^* \oplus \mathbb{R}\delta$ be the space of affine linear functions on V, where $\delta(v) = 1$ for all $v \in V$. Denote the gradient map by $D : \mathrm{Aff}(V) \twoheadrightarrow V^*$.

For $\phi \in \mathrm{Aff}(V)$, the orthogonal reflection $s_\phi : V \to V$ with respect to the affine hyperplane $V_\phi := \{v \in V | \phi(v) = 0\}$ is given by

$$s_\phi(v) := v - \phi(v)(D\phi)^\vee.$$

Define the translation map $t_{v'} : V \to V$ for $v' \in V$ by

$$t_{v'}(v) := v + v'.$$

We also denote s_ϕ and t_v for the corresponding transpositions acting on the space of functions on V, that is, $(s_\phi f)(v) := f(s_\phi(v))$ and $(t_{v'} f)(v) := f(t_{-v'}(v))$.

Set $\alpha_{ij} := \epsilon_i - \epsilon_j$ for $1 \leq i,j \leq k$. The subset $R_0 := \{\alpha_{ij} \,|\, 1 \leq i,j \leq k,\, i \neq j\}$ of $\mathrm{Aff}(V)$ forms the root system of type A_{k-1}. The Weyl group W_0 is generated by $\{s_\alpha\}_{\alpha \in R_0}$. We regard the set $R := R_0 + \mathbb{Z}(L\delta)$ as the affine root system of type $A_{k-1}^{(1)}$ with null roots $\mathbb{Z}(L\delta)$. The group W generated by $\{s_a\}_{a \in R}$ is called the *affine Weyl group of type* $A_{k-1}^{(1)}$. Any element of W is uniquely written in the form $wt_{L\beta}$ where $w \in W_0$ and β is an element of the coroot lattice $Q^\vee := \sum_{\alpha \in R_0} \mathbb{Z}\alpha^\vee$. In this sense we have $W = W_0 \ltimes (LQ^\vee)$. The gradient map $D : W \twoheadrightarrow W_0$ defined by $D(wt_{L\beta}) = w$ is a group homomorphism.

The *extended affine Weyl group* $\widehat{W}$ is generated by $\{s_a\}_{a\in R}$ and $\{t_{Lx}\}_{x \in X}$. Set

$$\pi := t_{Lv_1} s_1 \cdots s_{k-1}.$$

Then $\widehat{W}$ is generated by π and W.

Set $a_0 := -\alpha_{1k} + L\delta$ and $a_i = \alpha_{i,i+1}\,(1 \leq i < k)$. The set $\{a_0, \ldots, a_{k-1}\}$ gives a basis of R. Denote $R^\pm$ for the set of the associated positive and negative roots.

The length of $w \in W$ is defined by $\ell(w) := \#(R^+ \cap w^{-1}R^-)$. We abbreviate s_{a_i} by s_i $(0 \leq i < k)$. If $w = s_{i_1} \cdots s_{i_r}$ $(0 \leq i_1, \ldots, i_r < k)$ is a reduced expression, then $r = \ell(w)$ and $R^+ \cap w^{-1}R^- = \{s_{i_r} \cdots s_{i_{p+1}}(a_{i_p})\}_{p=1}^r$.

For $v \in V$, set

$$I(v) := \{a \in R^+ \,|\, a(v) < 0\}.$$

For $0 \le i < k$ and $v \in V$, we have $\#I(s_i v) = \#I(v) - 1$ if and only if $a_i(v) < 0$, and then $I(s_i v) = s_i(I(v) \setminus \{a_i\})$. Note that $I(v) = \emptyset$ if and only if v belongs to the closure of the fundamental chamber

$$\overline{C_+} := \{v \in V \mid a_i(v) \ge 0 \ (0 \le \forall i < k)\}.$$

For any $v \in V$, the orbit Wv intersects $\overline{C_+}$ at one point. Take a shortest element $w \in W$ such that $wv \in \overline{C_+}$. Then $I(v) = R^+ \cap w^{-1}R^-$, and hence the shortest element is uniquely determined for each $v \in V$. Denote it by w_v.

Lemma 2.1. *Suppose that $I(v_1) \subset I(v_2)$. Then $w_{v_2} = w_{w_{v_1}v_2} w_{v_1}$ and $\ell(w_{v_2}) = \ell(w_{w_{v_1}v_2}) + \ell(w_{v_1})$.*

Proof. Let $w_{v_1} = s_{i_1} \cdots s_{i_r}$ be a reduced expression. Then

$$I(s_{i_p} \cdots s_{i_r} v_2) = s_{i_p}(I(s_{i_{p+1}} \cdots s_{i_r} v_2) \setminus \{a_{i_p}\})$$

for $1 \le p \le r$ because $I(v_1) \subset I(v_2)$. Therefore $I(v_2) = I(v_1) \sqcup w_{v_1}^{-1} I(w_{v_1}v_2)$ and $\ell(w_{v_2}) = \ell(w_{w_{v_1}v_2}) + \ell(w_{v_1})$. Since $w_{w_{v_1}v_2} w_{v_1}$ moves v_2 into $\overline{C_+}$, it is equal to w_{v_2}. $\square$

3. Definition of Hamiltonian

Denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued functions on X by $F(X)$. For $1 \le i \le k$, define $d_i^{\pm} \in F(X)$ by

$$d_i^+(x) := \#\{1 \le p < k \mid \sum_{j=i}^{i+p-1} a_j(x) \in L\mathbb{Z}_{\le 0}\}, \tag{3.1}$$

$$d_i^-(x) := \#\{1 \le p < k \mid \sum_{j=i-p}^{i-1} a_j(x) \in L\mathbb{Z}_{\le 0}\}, \tag{3.2}$$

where the index j of simple root a_j is read modulo k. These functions have the following property:

Proposition 3.1. *For $x \in X, 1 \le i \le k$ and $0 \le j < k$, we have*

$$d_i^{\pm}(s_j x) = \begin{cases} d_i^{\pm}(x) & (i \ne j, j+1), \\ d_{j+1}^{\pm}(x) \pm \theta(a_j(x) = 0) & (i = j), \\ d_j^{\pm}(x) \mp \theta(a_j(x) = 0) & (i = j+1), \end{cases}$$

where $\theta(P) = 1$ or 0 if P is true or false, respectively.

Now we define the operator H acting on $F(X)$ by

$$H := \sum_{i=1}^{k} \beta^{d_i^+}(t_{v_i} - \alpha d_i^-),$$

where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}^\times$ are constants. Setting $\alpha = -1$ and $\beta = 1$, we have

$$H = \sum_{i=1}^{k} t_{v_i} - \sum_{\substack{1 \le i < j \le k \\ m \in \mathbb{Z}}} \theta(\alpha_{ij}(\cdot) + mL = 0).$$

It gives the discrete Hamiltonian $\tilde{H}$ (1.2) with periodic boundary condition where the system size is equal to L.

The operator H is W-invariant in the following sense. Set

$$X_{\mathrm{reg}} := X - \bigcup_{a \in R^+} V_a.$$

Proposition 3.2. *For $f \in F(X)$ and $w \in W$, it holds that $wHw^{-1}f = Hf$ on X_{reg}.*

Proof. Take a point $x \in X_{\mathrm{reg}}$ and $w \in W$. Define $\mu \in \mathfrak{S}_k$ by $(Dw)(v_i) = v_{\mu(i)}$ $(1 \le i \le k)$, then Proposition 3.1 implies that $d_i^\pm(w^{-1}x) = d_{\mu(i)}^\pm(x)$. Hence we have

$$(wHw^{-1}f)(x) = \sum_{i=1}^{k} \beta^{d^+_{\mu(i)}(x)}(f(wt_{-v_i}w^{-1}x) - \alpha d^-_{\mu(i)}(x) f(x)).$$

Since $wt_{-v_i}w^{-1} = t_{-(Dw)(v_i)} = t_{-v_{\mu(i)}}$, the right hand side is equal to $(Hf)(x)$. $\square$

4. Integral-reflection operators

4.1. Affine Hecke algebra.

Definition 4.1. The *affine Hecke algebra $\widehat{\mathcal{H}}$ of type GL_k* is the unital associative algebra with generators T_i $(1 \le i < k)$ and Y_i $(1 \le i \le k)$ satisfying

$$(T_i - 1)(T_i + \beta) = 0 \quad (1 \le i < k), \tag{4.1}$$

$$\begin{aligned} &T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \quad (1 \le i \le k-2), \quad T_iT_j = T_jT_i \quad (|i-j| > 1), \\ &Y_iY_j = Y_jY_i \quad (1 \le i, j \le k), \\ &Y_iT_j = T_jY_i \quad (j \ne i-1, i), \quad T_iY_{i+1}T_i = Y_i \quad (1 \le i < k). \end{aligned} \tag{4.2}$$

Set $\omega := Y_kT_{k-1}\cdots T_1$. Then $\omega T_i = T_{i-1}\omega$ $(1 < i < k)$ and $\omega^2 T_1 = T_{k-1}\omega^2$. The subalgebra $\mathcal{H}$ generated by T_i $(1 \le i < k)$ and $T_0 := \omega T_1 \omega^{-1}$ is called the *affine Hecke algebra of type $A_{k-1}^{(1)}$*.

4.2. **Integral-reflection operators.** Let us construct the integral-reflection operators acting on $F(X)$ from divided difference operators acting on polynomial ring following [6].

We identify the group algebra $\mathbb{C}[X]$ with the Laurent polynomial ring $\mathbb{C}[e^{\pm v_1}, \dots, e^{\pm v_k}]$. The extended affine Weyl group acts on $\mathbb{C}[X]$ by $w(e^x) := e^{wx}$ $(w \in \widehat{W}, x \in X)$.

Consider the operator $\check{T}_i\,(1 \le i < k)$ acting on $\mathbb{C}[X]$ defined by

$$\check{T}_i := s_i + \frac{\alpha e^{v_{i+1}} + 1 - \beta}{1 - e^{-\alpha_{i,i+1}}}(1 - s_i).$$

In [12] Lascoux and Schützenberger characterized divided difference operators acting on polynomial ring which satisfy the braid relation. They are parameterized by four parameters. The operators $\check{T}_i\,(1 \le i < k)$ are obtained by setting one of the parameters to zero, and they satisfy the quadratic relation (4.1) and the braid relation (4.2).

The group algebra $\mathbb{C}[X]$ acts on $F(X)$ by $e^x f := t_{-x} f$ [1]. Define a non-degenerate bilinear pairing $F(X) \times \mathbb{C}[X] \to \mathbb{C}$ by $(f, p) := (pf)(0)$. We consider the operator $Q_i\,(1 \le i < k)$ acting on $F(X)$ determined by $(Q_i f, p) = (f, \check{T}_i p)\,(\forall p \in \mathbb{C}[X])$. An explicit formula for Q_i is given as follows.

Definition 4.2. The *integral-reflection operator* $Q_i\,(1 \le i < k)$ acting on $F(X)$ is defined by

$$(Q_i f)(x) := \begin{cases} f(s_i x) + \displaystyle\sum_{j=1}^{a_i(x)} \left(\alpha f(s_i x + j a_i^\vee + v_{i+1}) + (1-\beta) f(s_i x + j a_i^\vee)\right) & (a_i(x) > 0), \\ f(x) & (a_i(x) = 0), \\ f(s_i x) - \displaystyle\sum_{j=1}^{-a_i(x)-1} \left(\alpha f(s_i x - j a_i^\vee + v_{i+1}) + (1-\beta) f(s_i x - j a_i^\vee)\right) & (a_i(x) < 0). \end{cases}$$

The operators $Q_i\,(1 \le i < k)$ satisfy the same quadratic relations and braid ones as $\check{T}_i\,(1 \le i < k)$ because the relations are left-right symmetric. Denote by $\check{\pi}$ the action of $\pi \in \widehat{W}$ on $\mathbb{C}[X]$. It satisfies $\check{T}_i \check{\pi} = \check{\pi} \check{T}_{i-1}\,(1 < i < k)$ and $\check{T}_1 \check{\pi}^2 = \check{\pi}^2 \check{T}_{k-1}$. From $(\pi^{-1} f, p) = (f, \check{\pi} p)$, we find that $Q_i\,(1 \le i < k)$ and π give $\widehat{\mathcal{H}}$-module structure on $F(X)$:

Proposition 4.3. *The assignment $T_i \mapsto Q_i\,(1 \le i < k)$ and $\omega \mapsto \pi^{-1}$ extends uniquely to a representation $\rho : \widehat{\mathcal{H}} \to \mathrm{End}_{\mathbb{C}} F(X)$ of the affine Hecke algebra of type GL_k.*

In the rest of this paper we make use of the restriction of ρ to the subalgebra $\mathcal{H}$.

5. Propagation operator

Set $Q_0 := \rho(T_0) = \pi^{-1} Q_1 \pi$. Let $w = s_{i_1} \cdots s_{i_m}$ be a reduced expression of $w \in W$ and set $Q_w := Q_{i_1} \cdots Q_{i_m}$. It does not depend on the choice of reduced expression of w.

Theorem 5.1. *For $f \in F(X)$, define $G(f) \in F(X)$ by*

$$G(f)(x) := (w_x^{-1} Q_{w_x} f)(x).$$

[1]This action is different from that in [6] where $e^x f := t_x f$.

If f is an eigenfunction of the operator $\sum_{i=1}^{k} t_{\epsilon_i}$ with eigenvalue $\lambda \in \mathbb{C}$, then $G(f)$ satisfies $HG(f) = \lambda G(f)$.

The following lemma plays a key role in the proof of Theorem 5.1.

Lemma 5.2. *Let $f \in F(X)$ and $x \in X$. Define $\sigma \in \mathfrak{S}_k$ by $(Dw_x)(v_i) = v_{\sigma(i)}$ $(1 \le i \le k)$. Then we have*

$$((t_{v_i} - \alpha d_i^+)G(f))(x) = ((t_{v_{\sigma(i)}} + (1-\beta) \sum_{j=1}^{d_i^+(x)} t_{v_{\sigma(i)+j}})Q_{w_x} f)(w_x x).$$

for $1 \le i \le k$. In the right hand side the index j of v_j is read modulo k.

Proof. In the proof we fix x and i, and set $y := x - v_i \in X$, $x' := x - \frac{1}{2}v_i \in \mathbb{Q} \otimes_{\mathbb{Z}} X$, $l = d_i^+(x)$ and $p = \sigma(i)$. For any $a \in R^+$, it holds that $a(x') = a(x) - (Da)(v_i)/2 = a(y) + (Da)(v_i)/2$. Since $|(Da)(v_i)| \le 1$, the two sets $I(x)$ and $I(y)$ are included in $I(x')$. Therefore $w_{x'} = w_{w_x x'} w_x = w_{w_y x'} w_y$ and $\ell(w_{x'}) = \ell(w_{w_x x'}) + \ell(w_x) = \ell(w_{w_y x'}) + \ell(w_y)$ from Lemma 2.1.

Let us write down $w_{w_x x'}$. Since $w_x x \in \overline{C_+}$ and $a(w_x x') = a(w_x x) - (Da)(v_p)/2$ for any $a \in R$, we have

$$I(w_x x') = \{a \in R^+ \,|\, a(w_x x) = 0,\ (Da)(v_p) > 0\}.$$

Now note that $l = d_p^+(w_x x)$ because w_x is shortest. If $z \in \overline{C_+}$, it holds that $d_j^+(z) = \max\{0 \le c \le k-1 \,|\, \sum_{r=j}^{j+c-1} a_r(z) = 0\}$. Therefore $I(w_x x') = \{s_p \cdots s_{p+j-1}(a_{p+j})\}_{j=0}^{l}$, where the index j of s_j and a_j is read modulo k. Thus we get

$$w_{w_x x'} = s_{p+l-1} \cdots s_{p+1} s_p. \tag{5.1}$$

Note that $a_{p+j}(w_x x) = 0$ for $0 \le j < l$.

Starting from the fact

$$I(w_y x') = \{a \in R^+ \,|\, a(w_y y) = 0,\ (Da)(v_q) < 0\},$$

where $(Dw_y)(v_i) = v_q$, we see that

$$w_{w_y x'} = s_{q-l'} \cdots s_{q-1}, \tag{5.2}$$

where $l' := d_i^-(y)$. The index j of s_j in the right hand side is read modulo k. Here note that $a_{q-j}(w_y y) = 0$ for $1 \le j \le l'$.

From (5.1) and (5.2), we find that

$$\begin{aligned} s_{q-l'} \cdots s_{q-1} w_y &= s_{p+l-1} \cdots s_{p+1} s_p w_x, \\ Q_{w_y} &= Q_{q-1}^{-1} \cdots Q_{q-l'}^{-1} Q_{p+l-1} \cdots Q_{p+1} Q_p Q_{w_x}. \end{aligned}$$

The first relation above implies that

$$w_y y = (s_{p+l-1} \cdots s_{p+1} s_p w_x)(x - v_i) = w_x x - v_{p+l}.$$

Therefore

$$G(f)(y) = (Q_{w_y}f)(w_y y) = (t_{v_{p+l}} Q_{p+l-1} \cdots Q_{p+1} Q_p Q_{w_x} f)(w_x x).$$

Using the commutation relation

$$t_{v_{j+1}} Q_j = Q_j t_{v_j} + \alpha + (1-\beta) t_{v_{j+1}}, \quad t_{v_{j'}} Q_j = Q_j t_{v_{j'}} \ (j' \neq j, j+1), \tag{5.3}$$

and $a_{p+j}(w_x x) = 0 \, (0 \leq j < l)$, we obtain the desired formula. □

Now let us prove Theorem 5.1. We fix $x \in X$, and let σ be the permutation given in Lemma 5.2. Identify the set $\{1, \ldots, k\}$ with $\mathbb{Z}/k\mathbb{Z}$, and decompose it into intervals of the form $[p, p+l] \, (1 \leq p \leq k, \, 0 \leq l \leq k-1)$ having the property

$$a_{p-1}(w_x x) > 0, \quad a_{p+j}(w_x x) = 0 \ (0 \leq j < l), \quad a_{p+l}(w_x x) > 0.$$

Take one interval $[p, p+l]$. Then $d^+_{\sigma^{-1}(p+j)}(x) = d^+_{p+j}(w_x x) = l - j$ and $d^-_{\sigma^{-1}(p+j)}(x) = d^-_{p+j}(w_x x) = j$ for $0 \leq j \leq l$. From Lemma 5.2 we have

$$\begin{aligned}
&\sum_{j=0}^{l} \beta^{d^-_{\sigma^{-1}(p+j)}} ((t_{v_{\sigma^{-1}(p+j)}} - \alpha d^+_{\sigma^{-1}(p+j)}) G(f))(x) \\
&= \sum_{j=0}^{l} \beta^j ((t_{v_{p+j}} + (1-\beta) \sum_{r=1}^{l-j} t_{v_{p+j+r}}) Q_{w_x} f)(w_x x) = \sum_{j=0}^{l} (t_{v_{p+j}} Q_{w_x} f)(w_x x).
\end{aligned}$$

The above relation holds on each interval, and we get

$$(HG(f))(x) = \sum_{i=1}^{k} (t_{v_i} Q_{w_x} f)(w_x x).$$

From (5.3), the operator $\sum_{i=1}^{k} t_{v_i}$ commutes with $Q_w \, (w \in W)$. Therefore if f is an eigenfunction of $\sum_{i=1}^{k} t_{v_i}$ with eigenvalue λ, it holds that $(HG(f))(x) = \lambda f(x)$. This completes the proof.

6. Bethe wave functions

Let us construct $\widehat{W}$-invariant eigenfunctions by means of the Bethe ansatz method:

Proposition 6.1. *Suppose that $\lambda = (\lambda_1, \ldots, \lambda_k) \in (\mathbb{C}^\times)^k$ is a solution of the system of algebraic equations*

$$\lambda_i^L = \prod_{\substack{j=1 \\ (j \neq i)}}^{k} \frac{\beta \lambda_i - \lambda_j - \alpha}{\lambda_i - \beta \lambda_j + \alpha} \quad (1 \leq i \leq k). \tag{6.1}$$

Define the function h_λ by

$$h_\lambda(x) = \sum_{\sigma \in \mathfrak{S}_k} \prod_{1 \leq i < j \leq k} (\beta \lambda_{\sigma(i)} - \lambda_{\sigma(j)} - \alpha) \prod_{i=1}^{k} \lambda_i^{-\epsilon_i(x)} \quad (x \in \overline{C_+}) \tag{6.2}$$

and $h_\lambda(wx) = h_\lambda(x)$ for any $w \in W$. Then h_λ is $\widehat{W}$-invariant and an eigenfunction of H with eigenvalue $\sum_{i=1}^{k} \lambda_i$.

Proof. Denote by $f_\lambda \in F(X)$ the function defined by the right hand side of (6.2) on the whole X. The function $g_\lambda(x) := \prod_{i=1}^{k} \lambda_i^{-\epsilon_i(x)}$ satisfies

$$Q_i g_\lambda = s_i g_\lambda + \frac{\alpha + \beta\lambda_{i+1}}{\lambda_i - \lambda_{i+1}}(s_i - 1)g_\lambda.$$

Hence $Q_i f_\lambda = f_\lambda$ for all $1 \le i < k$. Moreover it holds that $\pi f_\lambda = f_\lambda$ if $\{\lambda_i\}_{i=1}^{k}$ is a solution to (6.1). Therefore we get $G(f_\lambda)(x) = f_\lambda(w_x x) = h_\lambda(w_x x) = h_\lambda(x)$. Since $\sum_{i=1}^{k} t_{v_i} f_\lambda = (\sum_{i=1}^{k} \lambda_i) f_\lambda$, we find that h_λ is an eigenfunction of H with eigenvalue $\sum_{i=1}^{k} \lambda_i$. Since $\widehat{W}x \cap \overline{C_+} = \{\pi^n w_x x\}_{n\in\mathbb{Z}}$, we have $G(f_\lambda)(\pi x) = G(f_\lambda)(x)$. Hence h_λ is $\widehat{W}$-invariant. □

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